Probabilistic exact deletion and probabilistic no-signalling

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Abstract

In this work we show that one cannot use non-local resources for probabilistic signalling even if one can delete a quantum state with the help of probabilistic quantum deletion machine. Here we find that probabilistic quantum deletion machine is not going to help us in identifying two statistical mixture at remote location. Also we derive the bound on deletion probability from no-signalling condition.

1 Introduction:

Entanglement lies in the heart of Quantum mechanics. The key feature of the entangled states is the non-local correlations exhibited by them. This nonlocal correlations can not be explained by any kind of theories like local hidden variable theory [1] etc. A. Einstein, B.Podolsky and N.Rosen proved that measurement outcome of one system can instantaneously affect the result of the other [2]. This observation suggest that one can exploit the non local nature of the entanglement to send super-luminal signals. However it was shown that operators at space like separated distances commute, the average of the observable at distant site remains the same and does not depend on the operation carried out by other party [3,4]. Herbert argued that one can amplify the quantum state on one part of the system and performing measurements on many copies of that he can get the information about what the other party has done [5]. Later it was shown that one cannot clone a single quantum state , popularly known as 'no-cloning' theorem [6]. It was also shown that one cannot clone two non orthogonal quantum states [7]. Since deterministic exact cloning is ruled out by linearity of quantum theory, the possibility of super luminal signalling using non locality of the entangled states faded away. Various cloning machines which produce inaccurate copies [8,9,10], cannot be used for super

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luminal signalling. It was shown that the fidelity criterion based on no signalling is consistent with the fidelity criterion of inaccurate copying [11]. But it was shown by Duan and Guo [12] that a set of states chosen secretly can be exactly cloned by a probabilistic cloning machine if and only if the states are linearly independent. Hardy and Song found out a limit on the number of states that can be cloned by a probabilistic cloning machine by using the no signalling criterion [13]. Pati showed that one cannot send super luminal signals even probabilistically with the help of a probabilistic quantum cloning machine [14]. Complementary to 'no-cloning' theorem, Pati and Braunstein introduced a new concept of deletion of an arbitrary quantum state and shown that an arbitrary quantum states cannot be deleted. This is due to the linearity property of quantum mechanics.Quantum deletion [15] is more like reversible 'uncopying' of an unknown quantum state. The corresponding no-deleting principle does not prohibit us from constructing the approximate deleting machine [16].J. Feng et.al. [17] showed that each of two copies of non-orthogonal and linearly independent quantum states can be probabilistically deleted by a general unitary-reduction operation. Like universal quantum cloning machine, D'Qiu [18] also constructed a universal deletion machine but unfortunately the machine was found to be non-optimal in the sense of fidelity. A universal deterministic quantum deletion machine is designed in an unconventional way that improves the fidelity of deletion from 0.5 and takes it to 0.75 in the limiting sense [16]. Recently we showed that splitting of quantum information [19] is consistent with the principle of no-signalling [20]. Pati and Braunstein showed that if quantum states can be deleted perfectly one can send signals faster than the speed of light [21]. If a message can be transmitted instantaneously with a certain no-zero probability of success less than unit, it is known as probabilistic super-luminal signalling. It had already been seen that deterministic as well as probabilistic cloning machine and deterministic deletion machine does not allow super luminal signalling between two distant parties.

The organization of this work is as follows. In section 2, we show that one cannot send signals faster than light using probabilistic deletion machine. To show that we have consider a joint system of two singlet states shared between two distant partners Alice and Bob. Thereafter, the application of Bob's probabilistic deletion machine on his qubit leads to superluminal signalling if Alice prepare her particles in different basis because the resulting reduced density matrix describing Bob's system will be different in different basis. However this will not be true. since it is a known fact that probabilistic deletion machine deletes one copy of a quantum state from among two copies if and only if the states are linearly independent, but the states we have considered and fed into the probabilistic deletion machine are not all linearly independent so we arrive at a contradiction. In section 3, we obtain a bound on deletion probability using the no-signalling condition.

2 No-signalling using maximally entangled states

In this section, we show that signals cannot be send faster than light even with some non-zero probability, using the probabilistic deletion machine and non-local resources such as maximally entangled states.
Let us consider two maximally entangled states $|\chi\rangle_{12}$ and $|\chi\rangle_{34}$, each prepared in a singlet state shared by two distant partners, say Alice and Bob.
The singlet state $|\chi\rangle_{12}$ can be written in terms of the basis $\{|\psi_i\rangle, \overline{|\psi_i\rangle}\}$(i=1,2) onto which Alice might do a measurement and therefore the state $|\chi\rangle_{12}$ is given by

$$|\chi\rangle_{12} = \frac{1}{\sqrt{2}}(|\psi_i\rangle\overline{|\psi_i\rangle} - \overline{|\psi_i\rangle}|\psi_i\rangle) \tag{1}$$

Similarly, the entangled state $|\chi\rangle_{34}$ looks exactly the same as $|\chi\rangle_{12}$. The particles 1 and 3 possessed by Alice and particles 2 and 4 belongs to Bob.
The state representing the combined system $|\chi\rangle_{12} \otimes |\chi\rangle_{34}$ for arbitrary qubit basis $\{|\psi_i\rangle, \overline{|\psi_i\rangle}\}$(i=1,2) is given by

$$\begin{aligned} |\chi\rangle_{12} \otimes |\chi\rangle_{34} \;\; = \;\; & \frac{1}{2}(|\psi_i\rangle_1|\psi_i\rangle_3\overline{|\psi_i\rangle}_2\overline{|\psi_i\rangle}_4 + \overline{|\psi_i\rangle}_1\overline{|\psi_i\rangle}_3|\psi_i\rangle_2|\psi_i\rangle_4 - \overline{|\psi_i\rangle}_1|\psi_i\rangle_3|\psi_i\rangle_2\overline{|\psi_i\rangle}_4 - \\ & |\psi_i\rangle_1\overline{|\psi_i\rangle}_3\overline{|\psi_i\rangle}_2|\psi_i\rangle_4) \end{aligned} \tag{2}$$

A probabilistic quantum deleting machine (PQDM) which will delete one copy among two copies of a quantum state chosen from a set $S = \{|\psi_i\rangle, \quad i = 1, 2\}$ is defined by

$$U(|\psi_i\rangle|\psi_i\rangle|P_0\rangle) = \sqrt{p_i}|\psi_i\rangle|\Sigma\rangle|P_0\rangle + \sqrt{1-p_i}|\Phi_i\rangle|P_1\rangle \tag{3}$$

$$U(\overline{|\psi_i\rangle}\overline{|\psi_i\rangle}|P_0\rangle) = \sqrt{p_{\perp i}}\overline{|\psi_i\rangle}|\Sigma\rangle|P_0\rangle + \sqrt{1-p_{\perp i}}\overline{|\Phi_i\rangle}|P_1\rangle \tag{4}$$

$$U(|\psi_i\rangle\overline{|\psi_i\rangle}|P_0\rangle) = |X_1\rangle|P_2\rangle \tag{5}$$

$$U(|\psi_i\rangle\overline{|\psi_i\rangle}|P_0\rangle) = |X_2\rangle|P_3\rangle \tag{6}$$

where $|\Sigma\rangle$ denote some standard blank state and $|P_0\rangle$, $|P_1\rangle$, $|P_2\rangle$ and $|P_3\rangle$ are probe states to tell us whether deletion has been successful or not. Here $|P_1\rangle$, $|P_2\rangle$, $|P_3\rangle$are not necessarily orthogonal, but each of them are orthogonal to $|P_0\rangle$. Here $|\Phi_i\rangle, \overline{|\Phi_i\rangle}$ are the composite system of the input state and ancilla whereas $|X_1\rangle$ and $|X_2\rangle$ are in general pure entangled states of the non-identical inputs and the ancilla. We also note that $\{|\phi_i\rangle, \overline{|\phi_i\rangle}\}$ are not necessarily orthonormal. Here p_i is the probability of successful deletion of quantum states. The unitary operator U together with a measurement on the probing device P deletes one of the two copies with probability of success p_i.
To investigate the question of the possibility of the superluminal signalling with non-zero probability, we carry out the action of probabilistic quantum deletion machine on one part of a composite system (say, on the state of Bob's particle 4).
After operating probabilistic quantum deletion machine (3-6) on the state of Bob's particle 4, the combined system (2) reduces to

$$|\chi\rangle_{12} \otimes |\chi\rangle_{34}|A\rangle_6|P_0\rangle_5 \longrightarrow U_{245}(|\chi\rangle_{12}|\chi\rangle_{34}|A\rangle_6|P_0\rangle_5 =$$

$$\frac{1}{2}[(\sqrt{p_{\perp i}}|\psi_i\rangle_1|\psi_i\rangle_3|\overline{\psi_i}\rangle_2 + \sqrt{p_i}|\overline{\psi_i}\rangle_1|\overline{\psi_i}\rangle_3|\psi_i\rangle_2)\ |\Sigma\rangle_4|P_0\rangle_5$$
$$+(\sqrt{1-p_{\perp i}}\ |\overline{\Phi_i}\rangle_{24} + \sqrt{1-p_i}\ |\Phi_i\rangle_{24})\ |P_1\rangle_5)$$
$$+|\psi_i\rangle_1|\overline{\psi_i}\rangle_3|X_1\rangle_{24}|P_2\rangle_5 + |\overline{\psi_i}\rangle_1|\psi_i\rangle_3|X_2\rangle_{24}|P_3\rangle_5] \tag{7}$$

where the ancilla $|A\rangle_6$ is attached by Bob and the states $\{|\psi_i\rangle\}$ and $\{|\overline{\psi_i}\rangle\}$, (i=1,2) are linearly independent.
Alice may perform a measurement on her particles 1 and 3 in any one of the two basis $\{|\psi_1\rangle, |\overline{\psi_1}\rangle\}$ and $\{|\psi_2\rangle, |\overline{\psi_2}\rangle\}$.
If Alice finds her particles in the basis $\{|\psi_1\rangle, |\overline{\psi_1}\rangle\}$ then the reduced density matrix of the particles 2 and 4 in Bob's subsystem is given by

$$\rho_{24} = Tr_{1356}(\rho_{123456}) = \frac{1}{4}[p_1|\psi_1\rangle\langle\psi_1| + p_{\perp 1}|\overline{\psi_1}\rangle\langle\overline{\psi_1}|] \otimes |\Sigma\rangle\langle\Sigma| \tag{8}$$

On the other hand If Alice finds her particles in the basis $\{|\psi_2\rangle, |\overline{\psi_2}\rangle\}$ then the reduced density matrix of the particles 2 and 4 takes the following form

$$\rho_{24} = Tr_{1356}(\rho_{123456}) = \frac{1}{4}[p_2|\psi_2\rangle\langle\psi_2| + p_{\perp 2}|\overline{\psi_2}\rangle\langle\overline{\psi_2}|] \otimes |\Sigma\rangle\langle\Sigma| \tag{9}$$

It is clear from equations (4) and (5) that the two statistical mixtures describing the reduced density matrices of the particles 2 and 4 in Bob's subsystem is different and therefore one can conclude from here that superluminal signalling is possible i.e. the two different statistical mixtures of the same particles allowed Bob to distinguish two preparation stages by Alice. But it is a known fact that the superluminal signalling is not possible. The fallacy arises because we treat the four states $\{|\psi_1\rangle, |\overline{\psi_1}\rangle, |\psi_2\rangle, |\overline{\psi_2}\rangle\}$ as linearly independent states, which is not true. Since the states $\{|\psi_2\rangle, |\overline{\psi_2}\rangle\}$ can be written as

$$|\psi_2\rangle = \cos\theta|\psi_1\rangle + \sin\theta|\overline{\psi_1}\rangle \tag{10}$$
$$|\overline{\psi_2}\rangle = \sin\theta|\psi_1\rangle - \cos\theta|\overline{\psi_1}\rangle \tag{11}$$

so the set $S = \{|\psi_1\rangle, |\overline{\psi_1}\rangle, |\psi_2\rangle, |\overline{\psi_2}\rangle\}$ is linearly dependent and hence Bob's probabilistic quantum deletion machine cannot delete perfectly one copy from the two copies of all the four states taken from the set S with non-zero probability. Hence one can easily rule out the possibility of probabilistic superluminal signalling using probabilistic quantum deletion machine.

3 Bounds on deletion probability from no-signalling condition

A bipartite state can be written as

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}|u_k\rangle|v_k\rangle \tag{12}$$

where $|u_i\rangle$ are orthogonal basis for Alice's Hilbert space H_A and $|v_i\rangle$ are non-orthogonal and linearly independent basis states for Bob's Hilbert space H_B.
The combined state of two copies of $|\psi\rangle_{AB}$ is given by

$$|\psi\rangle_{AB} \otimes |\psi\rangle_{AB} = \frac{1}{N}[\sum_{k=1}^{N} |u_k u_k\rangle |v_k v_k\rangle + \sum_{k\neq l} |u_k u_l\rangle |v_k v_l\rangle] \tag{13}$$

After the application of probabilistic quantum deletion machine on the two copies of non-orthogonal basis states $|v_i\rangle$, the combined state reduces to

$$\begin{aligned} |\xi\rangle_{ABCD} &= \frac{1}{N}[\sum_{k=1}^{N}\{|u_k u_k\rangle(\sqrt{p_k}|v_k\rangle|\Sigma\rangle|P_0\rangle + \sqrt{1-p_k}|\phi_k\rangle|P_1\rangle)\} + \sum_{k\neq l, k<l} |u_k u_l\rangle|X_1\rangle|P_2\rangle \\ &+ \sum_{k\neq l, k>l} |u_k u_l\rangle|X_2\rangle|P_3\rangle] \end{aligned} \tag{14}$$

Now to calculate the success rate of deleting a copy perfectly using probabilistic quantum deletion machine, we define a basis in Bob's Hilbert space as

$$|\zeta_i\rangle = \langle u_i u_i|\xi\rangle_{ABCD} = \frac{1}{N}[\sqrt{p_i}|v_i\rangle|\Sigma\rangle|P_0\rangle + \sqrt{1-p_i}|\phi_i\rangle|P_1\rangle] \tag{15}$$

Taking the inner product of the two distinct basis, we get

$$\langle\zeta_i|\zeta_j\rangle = \frac{1}{N^2}[\sqrt{p_i p_j}\langle v_i|v_j\rangle + \sqrt{(1-p_i)(1-p_j)}\langle\phi_i|\phi_j\rangle] \tag{16}$$

Taking modulus both sides and using the inequality $|x+y| \leq |x| + |y|$, we get

$$N^2|\langle\zeta_i|\zeta_j\rangle| \leq \sqrt{p_i p_j}|\langle v_i|v_j\rangle| + \sqrt{(1-p_i)(1-p_j)}|\langle\phi_i|\phi_j\rangle| \tag{17}$$

Further using the inequalities $\sqrt{p_i p_j} \leq \frac{p_i+p_j}{2}$, $\sqrt{(1-p_i)(1-p_j)} \leq (1 - \frac{p_i+p_j}{2})$ and $|\langle\phi_i|\phi_j\rangle| \leq 1$, we have

$$\frac{p_i+p_j}{2} \leq \frac{1 - N^2|\langle\zeta_i|\zeta_j\rangle|}{1 - |\langle v_i|v_j\rangle|} \tag{18}$$

It is clear from equation (18) that the average success probability of deletion of a qubit depends on the number of the linearly independent states that can be deleted and also on the inner product of the non-orthogonal states belonging to Bob's Hilbert space.

4 Conclusion

In summary we can say that in this work we have shown that even though probabilistic quantum deleting machine can delete one of the two copies at the input port with a non-zero probability , but still it is not possible to send super-luminal signal with certain non-zero probability. Thus we see that the quantum theory is in total agreement with the special theory of relativity.

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6 Reference

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